210 Quantum Mechanics Test

In the following *show all your working*

1. Determine if the following functions are eigenfunctions of the p_x operator. If yes, identify the eigenvalue, if no, explain why not. **Show your working.**

(b) x^3

(c) e^{ikx}

5 marks

$\Omega f = \omega f$ $\Omega = \text{operator}$ $\omega = \text{eigenvalue}$ $f = \text{eigenfunction}$ $\Omega = p_x = \frac{-i}{h}\frac{d}{dx}$	$f = x^3 \quad \text{evaluate:}$ $\frac{h}{i}\frac{d}{dx}x^3 = \omega x^3$ $\frac{d}{dx}x^3 = 3x^2$ $\frac{h}{i}\frac{d}{dx}x^3 = \frac{h}{i}3x^2$ $\frac{3h}{i}x^2 \neq \omega x^3$	$f = e^{ikx} \quad \text{evaluate:}$ $\frac{h}{i}\frac{d}{dx}e^{ikx} = \omega e^{ikx}$ $\frac{d}{dx}e^{ikx} = ike^{ikx}$ $\frac{h}{i}\frac{d}{dx}e^{ikx} = \frac{h}{i}ike^{ikx}$ $hke^{ikx} = \omega e^{ikx}$ $\text{where } \omega = hk$

definition (**1 mark**)
maths (**1 mark each**)
x^3 is not an eigenfunction because the eigenvalue equation does not hold, the *function* is different on the LHS and RHS of the equation (**1 mark**)
e^{ikx} is an eigenvector because the equation does hold with an eigenvalue of hk. (**1 mark**)

This question is of a familiar format presented in lectures and problems.

2. **Discuss** the function below, **define** each of the variables, **identify** the named functions and quantum numbers, **describe** the functional components and **explain** their origins/relevance.

$$Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\; sin^2\theta \;\; e^{-i2\phi}$$

10 marks

Associated Legendre function gives 3D "orbital" forms and ℓ 1mark

$$\underbrace{Y_2^{-2}}_{\text{spherical harmonic}}\underbrace{(\theta,\phi)}_{\text{polar coords}} = \underbrace{\frac{1}{4}\sqrt{\frac{15}{2\pi}}}_{\text{normalisation factor}} sin^2\theta\; e^{-i2\phi}$$

1mark spherical harmonic

polar coords

normalisation factor 1 mark

particle on a ring solution gives m_ℓ 1mark

$Y_\ell^{m_\ell}$ here $\ell = 2$ and $m_\ell = -2$

m_ℓ = magnetic QN 1 mark

ℓ = angular QN

1 mark for components given until 10 marks is reached
θ and φ are the spherical polar variables/coordinates (**1 mark**)
$Y_l^{m_l}(\theta,\phi)$ is a spherical harmonic (**1 mark**)
and gives rise to the angular nature of AOs (**1 mark**)
l *angular quantum number* and l=2 (**1 mark**)

the l=2 is for a dAO **(1 mark)**
m_l *magnetic quantum number* and m_l=-2 **(1 mark)**
the coefficients in the front are the *normalisation* **(1 mark)**
normalisation provides the "magnitude", and ensures only 2e occupy each orbital, and that orbitals are comparable **(1 marks)**
$\sin^2\theta$ is an associated Legendre function (gives l QN) **(1 mark)**
θ gives the 3D angular shape of the orbitals**(1 mark)**
$e^{-i2\varphi}$ is the solution for a particle on a ring φ (gives the m_l QN) **(1 mark)**
φ gives the imaginary part of the wavefunction, and or is the decay feature **(1 mark)**

The form of this question was a direct "copy" of a problem given in the L7 problems, but not this particular wavefunction.

3. For "particle in a box" wavefunction given below

$$\psi(x) = N sin\left(\frac{nx\pi}{L}\right) \text{ with } n = 1, 2...$$

(a) **derive** an expression for the n=3 normalisation coefficient, **show your working.**
(b) **evaluate** N for the n=3 level for a box of length L=2nm, **show your working**
(c) **sketch** the n=3 level probability curve on a diagram of the box.

10 marks

normalisation requires $\int_{limits} \psi^*\psi d\tau = 1$	general expression **(1 mark)**
particle in a box n=3 level $\psi = N sin(\frac{3\pi x}{L})$	
$\tau \to x$	
ψ = real so $\psi^*\psi = \psi^2$	
limits are between 0 and L	
$\int_{limits} \psi^*\psi d\tau \to \int_0^L N^2 sin^2(\frac{3\pi x}{L})dx$	express the integral **(1 mark)**
evaluate $\int_0^L N^2 sin^2(\frac{3\pi x}{L})dx$	
we know $\int sin^2(2ax)dx = \frac{1}{2}x - \frac{1}{4a}sin(2ax)$	
where 2a $= \frac{3\pi}{L}$	evaluate A **(1 mark)**
$\int_0^L N^2 sin^2(\frac{3\pi x}{L})dx = N^2\left[\frac{1}{2}x - \frac{L}{6\pi}sin(\frac{3\pi x}{L})\right]_0^L$	
$\left[\frac{1}{2}x - \frac{L}{6\pi}sin(\frac{3\pi x}{L})\right]_0^L = \left[\frac{1}{2}L - \frac{L}{6\pi}sin(\frac{3\pi L}{L})\right] - \left[\frac{1}{2}0 - \frac{L}{6\pi}sin(\frac{3\pi 0}{L})\right]$	evaluate B **(1 mark)**
$= \left[\frac{1}{2}L - \frac{L}{6\pi}sin(3\pi)\right] - \left[0 - \frac{L}{6\pi}sin(0)\right]$	
but $sin(3\pi) = sin(0) = 0$	
$= \left[\frac{1}{2}L - 0\right] - [0 - 0]$	evaluate C **(1 mark)**
$= \frac{1}{2}L$	
evaluate $\int_0^L N^2 sin^2(\frac{3\pi x}{L})dx = 1$	
$N^2\frac{1}{2}L = 1$	final expression **(1 mark)**
$N = \sqrt{\frac{2}{L}}$	6 marks total

evaluating for L=2nm
(2 marks)

$$N = \sqrt{\frac{2}{L}}$$
$$\text{for L=2nm}$$
$$= \sqrt{\frac{2}{20 \times 10^{-10}}}$$
$$= \left(\frac{2}{2 \times 10^{-10}}\right)^{\frac{1}{2}}$$
$$= \left(\frac{2}{2}\right)^{\frac{1}{2}} \left(\frac{1}{10^{-10}}\right)^{\frac{1}{2}}$$
$$\text{where } \left(\frac{1}{10^{-10}}\right)^{\frac{1}{2}} = (10^{10})^{\frac{1}{2}} = 10^{\frac{10}{2}} = 10^{5}$$
$$= \sqrt{1} \times 10^{5}$$

express the probability on a infinite square well, three humps, all values positive
(2 marks)

The students have been given the evaluation of the general N as a class problem. Here we have asked for a particular n. This is a very small stretch from the notes. Evaluating the N for a specific length is new, drawing the energy diagram was part of the assignment.