L2: QM Operators

Problems

- evaluate the following equations and identify those that are eigenvalue equations

$$\Omega = x \text{ and } f = e^{kx}$$
$$\Omega = E_k \text{ and } f = e^{ikx}$$

an eigenvalue equation has $\Omega f(x) = \omega f(x)$ where ω is a constant

$$xe^{kx} = x(e^{kx})$$
$$\omega = x \text{ but x is not a number, this is not an eigenvalue equation}$$

$$E_k e^{ikx} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(e^{ikx})$$
$$\frac{d}{dx}(e^{ikx}) = ik(e^{ikx})$$
$$\frac{d^2}{dx^2}(e^{ikx}) = \frac{d}{dx}(ike^{ikx}) = (ik)^2(e^{ikx}) = -k^2(e^{ikx})$$
$$\text{used } i^2 = \sqrt{-1} \times \sqrt{-1} = -1$$
$$E_k e^{ikx} = -\frac{\hbar^2}{2m}(-k^2 e^{ikx}) = +\frac{\hbar^2 k^2}{2m}(e^{ikx})$$
$$\omega = \frac{\hbar^2 k^2}{2m} \text{ is a constant, therefor this is an eigenvalue equation}$$

- is cos(ax) an eigenfunction of the operators d/dx, or d^2/dx^2?

$$\text{eigenvalue equations satisfy } \Omega\psi = \omega\psi$$
$$\Omega = \frac{d}{dx} \qquad \psi = cos(ax)$$
$$\frac{d}{dx}cos(ax) = -asin(ax)$$
$$\text{does not have the form of an eighenvalue equation}$$

$$\text{eigenvalue equations satisfy } \Omega\psi = \omega\psi$$
$$\Omega = \frac{d^2}{dx^2} \qquad \psi = cos(ax)$$
$$\frac{d^2}{dx^2}cos(ax) = \frac{d}{dx} - asin(ax) = -a(acos(ax) = -a^2 cos(ax)$$
$$\text{does have the form of an eighenvalue equation with } \omega = -a^2 \text{ and } \psi = cos(ax)$$

- show that $xp_xf \neq p_xxf$ where $f=x^2$ What is the commutator? Do x and p_x commute?

$$(xp_x)x^2 = (x\frac{\hbar}{i}\frac{\partial}{\partial x})x^2 \qquad (p_x x)x^2 = (\frac{\hbar}{i}\frac{\partial}{\partial x})xx^2$$

$$= \frac{\hbar}{i}x\frac{\partial x^2}{\partial x} = \frac{\hbar}{i}x2x = \frac{2\hbar}{i}x^2 \qquad = \frac{\hbar}{i}\frac{\partial x^3}{\partial x} = \frac{\hbar}{i}3x^2$$

$$= \frac{2\hbar}{i}\frac{i}{i}x^2 = \frac{2\hbar i}{-1}x^2 \qquad = \frac{\hbar}{i}\frac{i}{i}3x^2$$

$$= -2\hbar i x^2 \qquad = -3\hbar i x^2$$

$$\text{thus } -2\hbar i x^2 \neq -3\hbar i x^2$$

$$\text{the commutator of two operators A and B is } [A,B] = AB - BA$$

$$[x,p_x]x^2 = -2\hbar i x^2 - -3\hbar i x^2 = \hbar i x^2$$

$$\text{thus } [x,p_x] = i\hbar$$

$$\text{to commute, two operators A and B must have } [A,B] = 0$$

$$\text{thus } x \text{ and } p_x \text{ do not commute}$$

- determine the commutator of the operators d/dx and x^2

$$[\frac{d}{dx},x^2]\psi = (\frac{d}{dx}x^2 - x^2\frac{d}{dx})\psi$$

$$= \frac{d}{dx}x^2\psi - x^2\frac{d}{dx}\psi$$

$$= (2x\psi + x^2\psi') - x^2\psi'$$

$$= 2x\psi + \cancel{x^2\psi'} - \cancel{x^2\psi'}$$

$$= 2x\psi$$

$$\text{operators A and B commute if } [A,B] = 0$$

$$\text{thus } \frac{d}{dx} \text{ and } x^2 \text{ do not commute}$$

$$\text{the commutator } [\frac{d}{dx},x^2] = 2x$$

- write out an expression for p(x,y,z) and $p(x,y,z)^2$ and thus determine the 3D kinetic energy operator, remembering that the dot product of two vectors:

$$\vec{a} = a_1x + a_2y + a_3z \text{ and } \vec{b} = b_1x + b_2y + b_3z$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$p_x^2 = (\frac{\hbar}{i}\frac{d}{dx})(\frac{\hbar}{i}\frac{d}{dx})$$

$$= \frac{\hbar^2}{i^2}\left(\frac{d^2}{dx^2}\right) \qquad E_k = \frac{p^2}{2m}$$

$$= -\hbar^2\left(\frac{d^2}{dx^2}\right) \quad \text{therefor} \quad = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}\right)$$

$$\vec{p} = p_x + p_y + p_z$$

$$\vec{p} = \frac{\hbar}{i}\nabla = \frac{\hbar}{i}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$$

∇ is the nabla or del

$$\vec{p} \cdot \vec{p} = p_x p_x + p_y p_y + p_z p_z$$

$$p^2 = -\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$= -\hbar^2\nabla^2$$

∇^2 is the laplacian or del squared

$$E_k = \frac{-\hbar^2}{2m}\nabla^2$$